High cohomology of arithmetic groups

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joint with

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K number field, $\mathcal{O}_K$ number ring

§1 Low cohomology

Borel stability (1974)

$\Gamma_n \underset{f.i.}{\leq} SL_n \mathcal{O}_K$, $i \ll n$

$H^i(\Gamma_n; \mathbb{Q})$ is computed and independent of n and Γ_n.

Homological stability (van der Kallen 1980)

$H^i(SL_n \mathcal{O}_K; \mathbb{Z})$, $H^i(GL_n \mathcal{O}_K; \mathbb{Z})$ is indep. of $n \gg i$.

Representation stability

$\Gamma_n(I) := \ker(SL_n \mathcal{O}_K \longrightarrow SL_n \mathcal{O}_K / I)$

Lee-Szczarba (1976) $H_1(\Gamma_n(p); \mathbb{Z})$

[LS76] $\cong \mathfrak{sl}_n \mathbb{F}_p \cong \mathbb{F}_p^{n^2-1}$ for $n \geq 3$.

Putman (2015), Church–Ellenberg–Farb–Nagpal (2017)

$$\mathrm{Ind}_{S_n}^{S_{n+1}} H_i(\Gamma_n(I);\mathbb{Z}) \twoheadrightarrow H_i(\Gamma_{n+1}(I);\mathbb{Z})$$

for $n \gg i$.

§2 Borel–Serre Duality (1973)

$\Gamma_n \underset{\text{f.i.}}{\leq} SL_n \mathcal{O}_K$

$$H^{vcd_n - i}(\Gamma_n;\mathbb{Q}) \cong H_i(\Gamma_n;\mathbb{Q}\otimes St_n K)$$

(Steinberg module $\to St_n K$)

e.g. $vcd_n = \binom{n}{2}$ for $\mathcal{O}_K = \mathbb{Z}$.

Cor $H^i(\Gamma_n;\mathbb{Q}) \cong 0$ if $i > vcd_n$.

Tits building $T_n(K)$ is simp cpx of flags of K^n, e.g. p-simplex:

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq K^n$$

Solomon–Tits (1969) $\tilde{H}_i(T_n(K);\mathbb{Z}) \cong 0$ unless $i = n-2$

Steinberg module $St_n(K) := \tilde{H}_{n-2}(T_n(K);\mathbb{Z})$ ↺ $SL_n K$

§3 Church–Farb–Putman (2014) conjecture

Conj $H^{\binom{n}{2}-i}(SL_n\mathbb{Z};\mathbb{Q}) \cong 0$ for $n \geq i+2$.
$\cong H_i(SL_n\mathbb{Z};\mathbb{Q}\otimes St_n\mathbb{Q})$

i=0 [LS76]
$\rightsquigarrow K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$
$\rightsquigarrow K_4(\mathbb{Z}) \cong 0$ (Rognes 2000)

i=1 Church–Putman (2017)
$\rightsquigarrow K_8(\mathbb{Z}) \cong 0$ (Dutour Sikiric et.al.)

i=2 Miller–P.–Putman–Wilson (in preparation)

Rem:
- $K_i(\mathbb{Z})$ is known for $i \not\equiv 0 \pmod 4$ by Voevodsky (2003) (and others)
- $K_{4i}(\mathbb{Z}) \cong 0$ for $i>0 \iff$ Kummer–Vandiver conj
- CFP-conj for $i \leq 4 \rightsquigarrow K_{12}(\mathbb{Z}) \cong 0$

(Quillen: $E^1_{pq} \cong H_q(GL_p\mathbb{Z}; St_p\mathbb{Q}) \overset{H_{p+q}}{\Longrightarrow} (\Omega^{\infty-1}_{12} K(\mathbb{Z}))$

BQ($\mathbb{Z}$)

§4 Other rings

Recall: · $\mathcal{O}_K$ Dedekind

- class group $cl(\mathcal{O}_K) \cong \widetilde{K}_0(\mathcal{O}_K)$ finite
- $|cl(\mathcal{O}_K)| = 1 \iff \mathcal{O}_K$ is PID

Non-PID: Church-Farb-Putman (2019)

$$\dim H^{vcd_n}(SL_n \mathcal{O}_K ; \mathbb{Q}) \geq (|cl(\mathcal{O}_K)| - 1)^{n-1}$$

PID, non-Euclidean

Generalized Riemann Hypothesis:

$\implies K = \mathbb{Q}[\sqrt{d}]$, $d \in \{-19, -43, -67, -163\}$

Miller-P.-Wilson-Yasaki (2020)

$$\dim H^{vcd_{2n}}(SL_{2n}(\mathcal{O}_{\mathbb{Q}[\sqrt{d}]}) ; \mathbb{Q}) \geq \begin{cases} 1 & d = -43 \\ 2^n & d = -67 \\ 6^n & d = -163 \end{cases}$$

Euclidean [LS76]

$H^{vcd_n}(SL_n \mathcal{O}_K ; \mathbb{Q}) \cong 0$ for $n \geq 2$ if $\mathcal{O}_K$ is Eucl.

Codim 1 Miller-Kupers-P.-Wilson

$H^{vcd_n - 1}(SL_n \mathcal{O}_K ; \mathbb{Q}) \cong 0$ for $n \geq 3$ if $\mathcal{O}_K = \mathbb{Z}[i]$ or $\mathbb{Z}\left[\frac{1+\sqrt{3}i}{2}\right]$

§5 Congruence subgroups

free of rank $3^{\binom{n}{2}}$

[LS76] : $H^{\binom{n}{2}}(\Gamma_n(3); \mathbb{Z}) \cong St_n \mathbb{F}_3$

Conj [LS76] :

$$H^{\binom{n}{2}}(\Gamma_n(p); \mathbb{Z}) \cong \tilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p); \mathbb{Z})$$

$\cong$ Borel-Serre

$H_0(\Gamma_n(p); St_n \mathbb{Q})$

$\cong$

$\tilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q}); \mathbb{Z})_{\Gamma_n(p)}$

Miller – P. – Putman

$$H^{\binom{n}{2}}(\Gamma_n(p); \mathbb{Z}) \longrightarrow \tilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p); \mathbb{Z})$$

is always surjective

is injective for $p \in 3$ or 5

is not injective for $p \geq 7$, $n \geq 2$.

Rem: I computed the ranks of $\tilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q})/\Gamma_n(5))$ for $n \leq 200$ in under 1 min.

Conj (Church-Farb-Putman 2014)

$H^{\mathrm{vcd}_n - i}(\Gamma_n(p);\mathbb{Z})$ should exhibit some kind of stabity.

Miller-Nagpal-P. (2020)

$$\mathrm{Ind}_{SL_n\mathbb{F}_p}^{SL_{n+1}\mathbb{F}_p} H^{\binom{n}{2}}(\Gamma_n(p);\mathbb{Z}) \twoheadrightarrow H^{\binom{n+1}{2}}(\Gamma_{n+1};\mathbb{Z})$$

$$\mathrm{Ind}_{SL_n\mathbb{F}_3}^{SL_{n+1}\mathbb{F}_3} H^{\binom{n}{2}-1}(\Gamma_n(3);\mathbb{Z}) \twoheadrightarrow H^{\binom{n+1}{2}-1}(\Gamma_{n+1}(3);\mathbb{Z})$$

for $n \geq 4$.

Conj (Miller-Nagpal-P. 2020)

$$\mathrm{Ind}_{SL_n\mathbb{F}_p}^{SL_{n+1}\mathbb{F}_p} H^{\binom{n}{2}-i}(\Gamma_n(p);\mathbb{Z}) \twoheadrightarrow H^{\binom{n+1}{2}-i}(\Gamma_{n+1}(p);\mathbb{Z})$$

for $n \gg i$.

§ 6 Methods

Borel-Serre $\longrightarrow$ Find "nice" resolutions of $\mathrm{St}_n K$

Generators:

$v_1, \dots, v_n$ a basis of K^n

$\rightsquigarrow [\![v_1, \dots, v_n]\!]$ apartment class of $\mathrm{St}_n K$

Solomon-Tits (1969) Apart cls generate $\mathrm{St}_n K$

$v_1, \dots, v_n$ a basis of $\mathcal{O}_K^n \subseteq K^n$

call $[\![v_1, \dots, v_n]\!]$ integral.

Ash-Rudolph (1976) Int. apart cl. gen. $\mathrm{St}_n K$ if $\mathcal{O}_K$ is Euclidean.

Proof Sketch:

- PB_n simp cpx of partial bases of $\mathcal{O}_K^n$
- If $\mathcal{O}_K$ Eucl., PB_n is $(n-2)$-ctd
- PB_n' partial bases that don't span $\mathcal{O}_K^n$

$$H_{n-1}(PB_n, PB_n') \xrightarrow{\partial} \tilde{H}_{n-2}(PB_n') \xrightarrow{K\text{-span}} \tilde{H}_{n-2}(\mathcal{T}_n(K))$$

$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{Z}[\text{unord. bases of } \mathcal{O}_K^n] & & \mathrm{St}_n K \\ (\text{oriented}) & & \end{array}$$

□

<u>Cor</u> $H^{\mathrm{vcd}_n}(SL_n \mathcal{O}_K ; \mathbb{Q}) \cong 0$ for $n \geq 2$

<u>Proof</u>: $\cong H_0(SL_n \mathcal{O}_K ; \mathbb{Q} \otimes \mathrm{St}_n K)$

$\cong (\mathbb{Q} \otimes \mathrm{St}_n K)_{SL_n \mathcal{O}_K}$

$[\![v_1, \ldots, v_n]\!] = A \cdot [\![v_2, -v_1, v_3, \ldots, v_n]\!]$

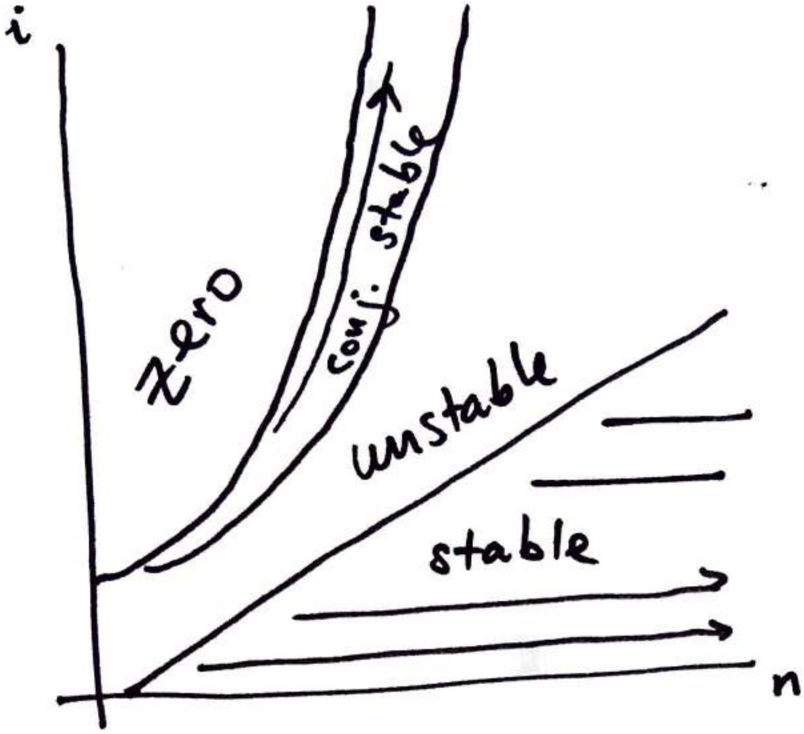
int. ap. cl. $\quad A \in SL_n \mathcal{O}_K$

$[\![v_2, -v_1, v_3, \ldots, v_n]\!] = [\![v_2, v_1, v_3, \ldots, v_n]\!]$

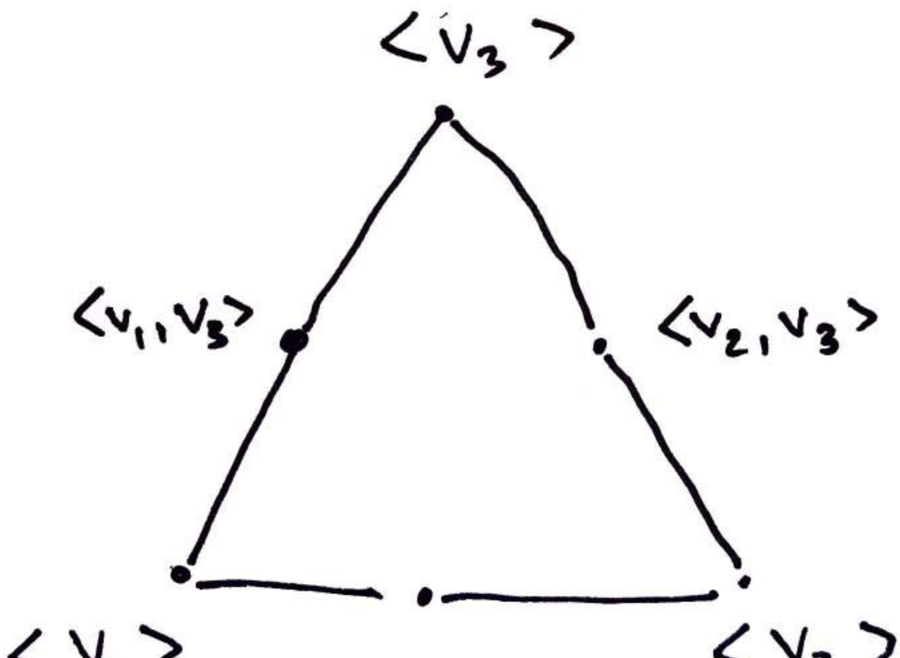
$= -[\![v_1, v_2, v_3, \ldots, v_n]\!]$

$\Rightarrow 2 \cdot [\![v_1, \ldots, v_n]\!] = 0$ in coinvariants □

$H^i(\Gamma_n)$



i
n
zero
conj. stable
unstable
stable



$\langle V_3 \rangle$
$\langle v_1, v_3 \rangle$
$\langle v_2, v_3 \rangle$
$\langle V_1 \rangle$
$\langle v_1, v_2 \rangle$
$\langle V_2 \rangle$