MIDWEST REPRESENTATION STABILITY RESEARCH MEETING PROBLEM SESSION

1. Rohit Nagpal's Question

Background. Let $\mathrm{St}_n(\mathbb{Z})$ denote the free abelian group on symbols: $[v_1, \ldots, v_n]$ with $v_1, \ldots, v_n$ a basis of $\mathbb{Z}^n$ subject to the relations:

i) $[v_1, \ldots, v_n] = sgn(\sigma)[v_{\sigma(1)}, \ldots, v_{\sigma(n)}]$,
ii) $[v_1, \ldots, v_n] = [-v_1, \ldots, v_n]$,
iii) $[v_1, v_2, \ldots, v_n] + [v_1 + v_2, v_1, \ldots, v_n] = [v_1 + v_2, v_2, \ldots, v_n]$.

Questions.

- Can we find an explicit subset of this generating set which is a basis?
- Can we find such a basis which is closed under multiplication by unit upper-triangular matrices?

Comments.

- This is a possible approach to proving the Church-Farb-Putman vanishing conjectures for $H^*(\mathrm{SL}_n(\mathbb{Z}))$.
- The analogous statement with $\mathbb{Z}$ replaced with a field is known and is part of the Solomon-Tits theorem.

2. Graham White's Question

Background. In the Kneser graph, the largest clique has size $\lfloor \frac{n}{2} \rfloor$ and the largest independent set has size $n - 1$. The Kneser graph is a finitely generated FI-graph.

Questions.

- For a finitely generated FI-graph, how do the clique size and largest independent set grow?
- Can one make similar statements about other invariants similar to clique size or size of a largest independent set?
- Is there a description of the set of subgraphs which realize the largest clique or independent set that has an eventually uniform description in the spirit of FI-modules?

3. Nate Harman's Question

Background. For n sufficiently large, $\mathrm{GL}_n(Z)$ and $\mathrm{Aut}(F_n)$ have property (T). Let $\mathsf{VIC}(\mathbb{Z})$ be the category of finite rank free $\mathbb{Z}$-modules with morphisms split linear injections with choice of complement. Let $U\,\mathrm{Aut}(F)$ denote the analogue for $\mathrm{Aut}(F_n)$. That is, $U\,\mathrm{Aut}(F)$ is the category of finite rank free groups with morphisms split injective group homomorphisms and choice of complement.

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Questions.

- Can one formulate and then prove property (T) for the categories $\mathsf{VIC}(\mathbb{Z})$ and $U\operatorname{Aut}(F)$?
- Is the correct notion of property (T) for these types of categories a stable notion? That is, should we only care about tails of $\mathsf{VIC}(\mathbb{Z})$-modules and $U\operatorname{Aut}(F)$-modules when formulating an analogue property (T)?

4. Benson Farb's Question

Background. Let V be a finitely generated FI-module. Then there are character polynomials

$$\chi_{V_n}(\sigma) = P(x_1, \ldots, x_r)$$

that compute characters in the stable range.

Questions.

- Can we compute these in examples such as for V the cohomology of ordered configuration spaces of surfaces?
- Is there an understandable space with an action of S_∞ that encodes the representation stable portion of the cohomology of a family of spaces exhibiting representation stability?

5. Jordan Ellenberg's Question

Background. Let G be a finite group. Let $V_n = \mathbb{Z}[G^n]$. There is an action of the braid group on V_n given by $\sigma_j(g_1, \ldots, g_n) = (g_1, \ldots g_{j-1}, g_j g_{j+1} g_j^{-1}, g_j, g_{j+2}, \ldots, g_n)$. The homology groups $H_i(Br_n; V_n)$ agree with those of a certain moduli space of branched covers with monodromy in G (a.k.a a Hurwitz space). For G replaced with certain conjugacy classes, Ellenberg–Venkatesh–Westerland proved that $H_i(Br_n; V_n)$ stabilizes as n tends to infinity. If you replace the braid group Br_n with pure braid group, then $\{H_i(PBr_n; V_n)\}_n$ has the structure of an FI-module. In work in progress, Ellenberg has likely shown that $\{H_i(PBr_n; V_n)\}_n$ is presented in finite degree.

Questions.

- Can one prove that the presentation degree of $\{H_i(PBr_n; V_n)\}_n$ is bounded by a linear function in i?

Comments.

- An affirmative answer to this question would likely have number-theoretic applications in a similar spirit to those of the original Ellenberg–Venkatesh–Westerland project.
- The coefficient system V_n grows exponentially fast so it is not "polynomial."

6. John Wiltshire-Gordon's Question

Background. Let $Conf_n(X)$ denote the configuration space of n ordered points in X. Let Y denote the cone on 3 points. Let Z denote the 1-skeleton of the 3-simplex. We have $Conf_2(\mathbb{R}^3) \simeq S^2$, $Conf_2(Y) \simeq S^1$, $Conf_2(Z) \simeq S^2$, and $Conf_2(Y \times Y) \simeq S^3$. These homotopy equivalences are equivariant with respect to the antipodal action on S^n and the usual S_2 action on $Conf_2(X)$ permuting the two points.

Question.

- Can we use these equivalences to find obstructions to embedding spaces into other spaces?

7. Wee Liang Gan Question

Background. Consider a short exact sequence of FI-groups:

$$1 \to A \to B \to C \to 1.$$

The group homology $H_i(A)$, $H_i(B)$, and $H_i(C)$ will have the structure of FI-modules.

Question.

- If $H_i(B)$ and $H_i(C)$ are presented in finite degree of all i, is $H_i(A)$ presented in finite degree?

8. Jeremy Miller's Question

Background. Let $\mathrm{St}_n(K)$ denote the Steinberg module of a field K. Let $\mathcal{O}$ denote the ring of integers in K. Let $[L_1, \ldots, L_n]$ denote the apartment class (a.k.a. modular symbol) associated with a direct sum decomposition of K^n into lines. For $\mathcal{O}$ Euclidean, the Ash-Rudolph theorem says that $\mathrm{St}_n(K)$ is generated by apartment classes $[L_1, \ldots, L_n]$ with:

$$\mathcal{O}^n = (\mathcal{O}^n \cap L_1) \oplus \ldots \oplus (\mathcal{O}^n \cap L_n).$$

Questions.

- For $\mathcal{O}$ not necessarily Euclidean, is $\mathrm{St}_{2m}(K)$ generated by apartment classes $[L_1, \ldots, L_{2m}]$ with:

$$\mathcal{O}^{2m} = \big(\mathcal{O}^{2m} \cap (L_1 \oplus L_2)\big) \oplus \ldots \oplus \big(\mathcal{O}^{2m} \cap (L_{2m-1} \oplus L_{2m})\big)?$$

- What is the correct generalization for m odd?

Comments.

- This is equivalent to the statement that

$$\mathrm{Ind}_{(\mathrm{GL}_2(\mathcal{O}))^m}^{\mathrm{GL}_{2m}(\mathcal{O})} \mathrm{St}_2(K) \to \mathrm{St}_{2m}(K)$$

  is surjective.
- This would let you compute the top rational cohomology group of quadratic imaginary number rings which are PIDs but not Euclidean.

9. Eric Ramos's Question

Background. Let $n > k$. Let $X_{n,k} = \{(\ell_1, \ldots, \ell_n) | \sum \ell_i = \mathbb{C}^k\} \subseteq \mathbb{C}P^{k-1}$. There is a stabilizabtion map $X_{n,k} \to X_{n+1,k+1}$ given by

$$(\ell_1, \ldots \ell_n) \mapsto (\ell_1, \ldots \ell_n, span(e_{k+1})).$$

This map and the natural symmetric group actions give the sequence $k \mapsto X_{k+i,k}$ the structure of an FI-space. It is known that the sequence $\{H_j(X_{k+i,k})\}_k$ exhibits multiplicity stability.

Questions.

- Can we give an easier proof of this multiplicity stability using FI-modules?
- Is FI the correct category to consider?

10. Peter Patzt's Question

Background. Bykovskii constructed a presentation of $\mathrm{St}_n(\mathbb{Z})$ by $\mathbb{Z}[\mathrm{GL}_n(\mathbb{Z})]$-modules which are flat after inverting finitely many primes. See Rohit Nagpal's question for a description.

Question.

- Can one find a partial resolution of $\mathrm{St}_n(\mathbb{Z})$ in the spirit of Bykovskii?

Comment.

- Church–Putman used Bykovskii's presentation to show vanishing of the codimension 1 homology of $\mathrm{SL}_n(\mathbb{Z})$. A longer partial resolution could be used to show vanishing of higher codimension cohomology.

11. Alexander Kupers' Question

Background. For a finite set I, let $Emb(I, M)$ be the space of embeddings of I into M. Let $ToHoFib_{I\subset\{1,\dots,k\}} Emb(I, M)$ denote the total homotopy fiber. For example, for $k = 2$, this is

$$hofib\Big(hofib\big(PConf_2(M) \to M\big) \to hofib\big(M \to *\big)\Big).$$

Question.

- Can we compute $ToHoFib_{I\subset\{1,\dots,k\}} Emb(I, M)$?

Comment.

- These spaces appear in embedding calculus towers.