STABILITY IN TOPOLOGY, ARITHMETIC, AND REPRESENTATION THEORY – PROBLEM SESSION

Jeremy Miller:

Theorem (Galatius–Randal-Williams [GRW])**.** *Let M and N be compact smooth even-dimensional manifolds of dimension at least 6. In a stable range, $H_i(B\operatorname{Diff}(M);R) \cong H_i(B\operatorname{Diff}(N);R)$ if $\chi(M)/\chi(N)$ is a unit in R.*

This stability pattern is similar to stability patterns for configuration spaces [BM14, CP15]. Another stability pattern for configuration spaces is stability periodicity.

Theorem (Nagpal [Nag15, CP15, KM16])**.** *In a stable range, $H_i(\operatorname{Conf}_k(M);\mathbb{F}_p) \cong H_i(\operatorname{Conf}_j(M);\mathbb{F}_p)$ if p divides $k-j$.*

Question. *For M and N compact smooth manifolds, is it true that $H_i(B\operatorname{Diff}(M);\mathbb{F}_p) \cong H_i(B\operatorname{Diff}(N);\mathbb{F}_p)$ if p divides $\chi(M)-\chi(N)$ for i in a stable range?*

Dan Ramras/ Andrew Putman:

Theorem (Ramras–Stafa [RS])**.** *Let G be a Lie group. The sequence $\{H_i(\operatorname{Hom}(\mathbb{Z}^n, G)_1;\mathbb{Q})\}_n$ is representation stable as symmetric group representations.*

Here the subscript 1 denotes the connected component of the trivial representation.

Question. *Does the homology of $\operatorname{Hom}(\mathbb{Z}^n, \operatorname{Diff}(M))_1$ exhibit any form of stability?*

Alexander Kupers:

Given a ring R, let $T_n(R)$ denote the realization of the poset of non-trivial proper summands of R^n. Let $T_n^2(R)$ denote the subcomplex of $T_n(R) * T_n(R)$ of pairs of flags

$$(V_0 < \ldots < V_p) * (W_0 < \ldots < W_q)$$

such that for all i and j, $V_i \cap W_j$ is a summand and $V_i + W_j$ is a summand. Let $\operatorname{St}_n^{E_2}(R)$ denote $H_{2n-3}(T_n^2(R))$. Note that for R a field, $T_n^2(R) = T_n(R) * T_n(R)$ and $\operatorname{St}_n^{E_2}(R) \cong \operatorname{St}_n(R) \otimes \operatorname{St}_n(R)$.

Question. *If R is a semi-local ring with infinite residue fields. Is*

$$(\operatorname{St}_n^{E_2}(R))_{\operatorname{GL}_n(R)} \cong \mathbb{Z}?$$

Galatius–Kupers–Randall-Williams [GKRW] proved this for R a field. Semi-local rings are interesting as they map to fields and possibly could allow one to compare different fields via zig-zags.

Andrew Putman:

Let $\operatorname{St}_n(R)$ denote $\tilde{H}_{n-2}(T_n(R))$.

Theorem (Classical)**.** *If $\mathbb{F}$ is a finite field, then $\operatorname{St}_n(\mathbb{F}) \otimes_{\mathbb{Z}} \mathbb{C}$ is an irreducible $\operatorname{GL}_n(\mathbb{F})$-representation.*

Theorem (Galatius–Kupers–Randal-Williams [GKRW])**.** *If $\mathbb{F}$ is an infinite field, then $\operatorname{St}_n(\mathbb{F}) \otimes_{\mathbb{Z}} \mathbb{C}$ is an indecomposable $\operatorname{GL}_n(\mathbb{F})$-representation.*

Question. *Is $\operatorname{St}_n(\mathbb{F}) \otimes_{\mathbb{Z}} \mathbb{C}$ even irreducible if $\mathbb{F}$ is infinite?*

Note. Also see:
https://mathoverflow.net/questions/228359/is-the-steinberg-representation-always-irreducible.

Andrew Putman:

Theorem (Gunnells [Gun00])**.** *The virtual dualizing module (Steinberg module) of* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *is generated by a symplectic analogue of integral apartments in the sense of Ash–Rudolph* [AR79]*.*

Question. *Is there a presentation of the Steinberg module* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *analogous to Bykovskiĭ's* [Byk03] *presentation of the Steinberg module of* $\mathrm{SL}_n(\mathbb{Z})$*?*

Note. One can use the fact that the simplicial complex of partial bases of $\mathbb{Z}^n$ is highly connected to prove that the Steinberg module of $\mathrm{SL}_n(\mathbb{Z})$ is generated by integral apartments [CFP19, Theorem A]. Similarly, one might expect that high connectivity of complex of isotropic partial bases of $\mathbb{Z}^{2g}$ would imply Gunnells' theorem. Instead one needs to prove high connectivity of the complex of *augmented* isotropic partial bases of $\mathbb{Z}^{2g}$ considered in [Put07, Section 6.4]. One might expect that a similar complex with even more augmentations could be used to establish a presentation of the Steinberg module $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Alexander Kupers:

Let D_n denote the little n-disks operad. Given an operad $\mathcal{O}$, let $\mathrm{HoAut}(\mathcal{O})$ denote the (appropriately derived) space of homotopy automorphisms of $\mathcal{O}$ viewed as an operad.

Question. *What is* $\pi_0(\mathrm{HoAut}(D_n))$*?*

The identity and reflection gives elements of $\pi_0(\mathrm{HoAut}(D_n))$ but possibly there are other elements. An answer to this question could potentially have applications in embedding calculus.

Note. The answer is known for the following cases/variants:

- For $n \leqslant 2$,
- After rationalizing the operads,
- After pro-finitely completing the operad when $n = 2$.

Andrew Putman and Andrew Snowden:

For R a ring with a G action, a G-ideal is an ideal that is closed under the action of G. A G-ideal P is called G-prime if for all G-ideals I, J, if $IJ \subset P$, then $I \subset P$ or $J \subset P$.

Question. *Classify the* G*-primes of* $(S_\infty \wr \mathbb{Z}/2) \circlearrowright \mathbb{F}[x_1, x_1^{-1}, \dots]$.

Andrew Snowden:

Theorem (Nagpal–Snowden)**.** *Let* b *be the ideal in* $\mathbb{F}[t_1, \dots, t_n, u]$ *generated by the* $(t_i - u)^k$ *for* $1 \leqslant i \leqslant n$ *and fixed* k*. Let* $a = b \cap \mathbb{F}[t_1, \dots, t_n]$*. The elements* $(t_i - t_j)^{2k-1} \in a$ *for* $1 \leqslant i < j \leqslant n$ *generates* a*.*

Question. *Is there a more conceptual proof of this statement?*

Note. This shows that the ideal generated by the $(t_i - t_j)^{2k-1}$ for $1 \leqslant i < j \leqslant n$ is G-prime. A more conceptual proof might prove it in positive characteristic.

Aaron Landesman:

Let $S^{(n)}$ denote the moduli stack parameterizing geometrically integral degree n genus 1 curves in $\mathbb{P}^{n-1}$. Then the n-Selmer stack $\underline{\mathrm{Sel}}^d_{n,\mathbb{C}}$ of height d is the stack of degree d maps from $\mathbb{P}^1_{\mathbb{C}} \to S^{(n)}$.

Theorem (Landesman)**.** $H_0(\underline{\mathrm{Sel}}^d_{n,\mathbb{C}}; \mathbb{Q})$ *stabilizes for* $d \geqslant 2$*. (It is* $(\sum_{s|n} s)$*-dimensional.)*

Question. *Does* $H_i(\underline{\mathrm{Sel}}^d_{n,\mathbb{C}}; \mathbb{Q})$ *stabilize for* $d \gg i$*?*

Note. This has applications to the conjectures of Bhargava–Shankar and Poonen–Rains in arithmetic statistics in a function field setting.

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